Qualifying Exam—Numerical ODE 05/2022

Show intermediate results at all steps!

1. For ODE $\dot{x} = f(x)$ with f satifying Lipshcitz condition with bounded 2nd order derivative, prove the global convergence of the Euler scheme::

$$x_{n+1} = x_n + kf(x_n).$$

2. Derive the absolute stability region for the backward Euler scheme.

3. Consider the homogeneous ODE $\dot{u}(t) = f(u)$, show that the Local Truncation Error of the following midpoint method is of order $O(k^3)$:

$$U^{n+1} = U^n + kf\left(U^n + \frac{1}{2}kf(U^n)\right).$$

4. Ilustrate stability of the Trapezoidal method for the Harmonic oscillator:

$$\frac{dz}{dt} = Az, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad and \quad z = (q, v)^T.$$

5. Consider the Hamiltonian dynamics:

$$\frac{dq}{dt} = \nabla_p H(q,p), \quad \frac{dp}{dt} = -\nabla_q H(q,p) \tag{1}$$

Show that the implicit Euler-B scheme is symplectic

$$q^{n+1} = q^n + \Delta t \nabla_p H(q^n, p^{n+1}), \quad p^{n+1} = p^n - \Delta t \nabla_q H(q^n, p^{n+1}).$$

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6. Show that the implicit mid-point method

$$z^{n+1} = z^n + \Delta t J \nabla_z H\left(\frac{z^{n+1}+z^n}{2}\right), \quad z = (q,p)^T$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, is time reversible for solving Hamiltonian dynamics (1).

7. Formulate the Stomer-Verlet algorithm for equation (1) when H is separable such that $H = T(p) + V(q)$ and show its Leap-frog formulation.